Extremal points for polynomial interpolation
The constructive approach

Jean-Paul Calvi

A. The very basics on polynomial interpolation

Definition 1 — . $\mathbb{P}^d(\mathbb{C}^N)$ is the space of complex polynomials of (total) degree at most d. Its dimension is $N = \binom{N+d}{d}$.

Definition 2 — . A subset $A \subset \mathbb{C}^n$ of cardinality N is said to be **regular** if for every function f defined on A there exists a unique polynomial $p \in \mathbb{P}^d(\mathbb{C}^n)$ such that $p = f$ on A. In that case, we write

$$p = \mathbf{L}[A, f]$$

and call p the **Lagrange interpolation polynomial** of f at A.

Fact 1 — . To compute $\mathbf{L}[A, f]$ you just need to solve a $N \times N$ linear system. If $A = (a_1, \ldots, a_N)$, we have the **Lagrange interpolation formula**

$$\mathbf{L}[A, f](z) = \sum_{i=1}^{N} f(a_i)\ell_i(z) \quad \text{where} \quad \ell_i(z) = \frac{\mathbf{VDM}(a_1, \ldots, a_{i-1}, z, a_{i+1}, \ldots, a_N)}{\mathbf{VDM}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_N)},$$

and $\mathbf{VDM}(a_1, \ldots, a_N)$ is a **Vandermonde determinant**, that is the determinant of the matrix whose entries are the monomials $w \to w^\alpha$ of degree $\leq d$ evaluated at the a_i's.

Fact 2 — . ... Multivariate vandermondians are not easy to deal with ...

B. The natural problems to solve

Problem 1 — . Given a natural class F of functions, find points such that $\mathbf{L}[A, f]$ correctly approximate elements of F.

Problem 2 — . Given a natural family A of points, find functions that are correctly approximated by Lagrange interpolation polynomials at A.

Comment 1 — . When we say that we want to find good points,

- we are not saying that we want to give a property which is **equivalent** to be good points,
- we are saying that we want to produce (with the help of a computer) a **specific list of numbers** that can be used for real computations (with the help of a computer).

C. THE TOOLS AT OUR DISPOSAL

It depends very much on the dimension...

In the one dimensional case a miracle occurs ...

FACT 3 — . The error between the univariate Cauchy kernel $(1/(\cdot - z))$ and its interpolation polynomial at $A = (a_1, \dots, a_{d+1})$ is given by

$$\frac{(w-a_1)\dots(w-a_{d+1})}{(z-a_1)\dots(z-a_{d+1})}.$$

Studying approximation properties of analytic functions by interpolation polynomials comes to :

$\longrightarrow$ study the behaviour of discrete potentials like $\sum_{i=1}^{d+1} \log|z-a_i|$

$\longrightarrow$ exploit results of classical plane potential theory.

For instance, loosely speaking, the basic result is

$\longrightarrow$ *good points for Lagrange interpolation of analytic functions on a (regular polynomially convex) plane compact sets are those that discretize the equilibrium measure of the compact set.*

$\longrightarrow$ Famous examples include **Fekete points** and **Leja points** (to be considered later)

COMMENT 2 — . All of this very very rarely leads to **lists** of good interpolation points... Yet it provides possible starting points for the computation of such lists.

... but in the multivariate are (currently) are left with the sole ***Lebesgue inequality****...*

D. The Lebesgue inequality

Fact 4 — . If f is any continuous function defined on K and $A = (a_1, \dots, a_N)$ are interpolation points for degree d then

$$\|f - \mathbf{L}[A, f]\|_K \leq (1 + \Delta(A))\mathrm{dist}(f, \mathbb{P}^d(\mathbb{C}^n)),$$

where $\Delta(A)$ is the norm of the linear operator $f \to \mathbf{L}[A, f]$ known as **the Lebesgue constant** for A (and K), we have $\Delta(A) = \left\|\sum_{i=1}^N |\ell_i|\right\|_K$.

Fact 5 — . Given a *reasonable* compact set in $\mathbb{R}^n$ or $\mathbb{C}^n$, Lebesgue inequality suggests to look for sequences of interpolation points $A^d = (a_1^d, \dots, a_N^d)$ for interpolation in $\mathbb{P}^d(\mathbb{C}^n)$ - here N changes with d - such that

$\longrightarrow$ $(\star)$ $\Delta(A^d)$ grows sub-exponentially as $d \to \infty$.

- Because, in view of the Siciak-Bernstein-Walsh theorem this implies good approximation results for analytic functions.

$\longrightarrow$ $(\star\star)$ $\Delta(A^d)$ grows polynomially as $d \to \infty$, say $\Delta(A^d) \approx d^\alpha$.

- Because, in view of Jackson's theorem this implies good approximation results for differentiable functions of class $C^{\alpha+1}$.

$\longrightarrow$ $(\star\star\star)$ $\Delta(A^d)$ grows sub-polynomially as $d \to \infty$.

- Because, in view of Jackson's theorem this implies good approximation results for differentiable functions of class C^1 and we cannot expect better …

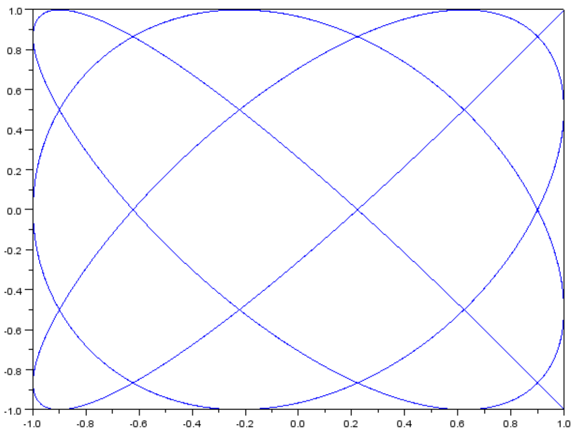
E. …BUT THE LEBESGUE INEQUALITY IS NOT ENOUGH AND WE NEED SOMETHING ELSE

…we may wait for another miracle…

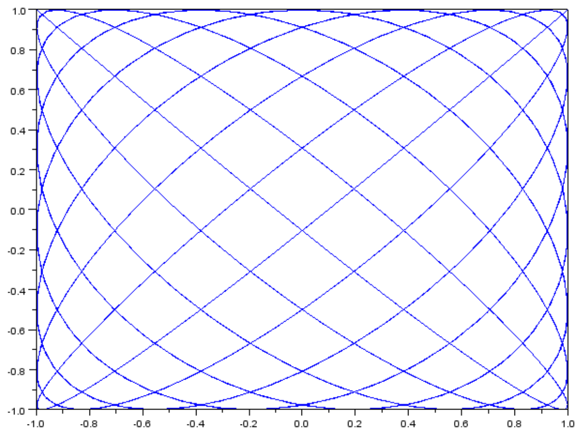
Miracles do indeed occur… but they are reserved to true believers…

THEOREM 1 (The Padua points miracle (Bos, Caliari, de Marchi, Vianello)) — . The double points of the curve $(\cos(n\theta), \cos((n+1)\theta)$ togther with the points on the boundary form a regular set for degree $d = n-2$ in $[-1,1]^2$ — called PADUA_d — and

$$\Delta(\text{PADUA}_d) \approx \log^2(d).$$



PADUA_4



PADUA_{13}

COMMENT 3 — . Padua points can be very easily computed as well as the corresponding interpolation polynomials.

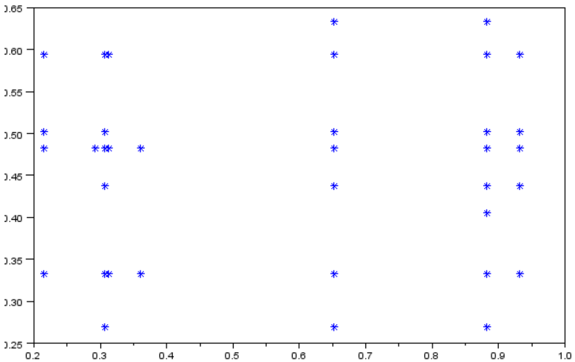
. . . Skeptics can't rely on miracles . . .

. . . they must come back to the univariate case . . .

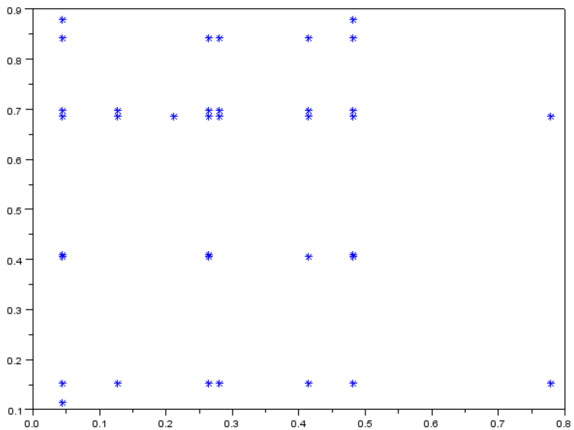
FACT 6 — . An easy way of obtaining a bivariate regular set starting from two univariate ones is the **Biermann intertwining process**.

$$\mathbf{A} = (a_0, \ldots, a_d),\ \mathbf{B} = (b_0, \ldots, b_d) \longrightarrow \mathbf{A} \oplus \mathbf{B} = ((a_i, b_j), i + j \leq d).$$

Her are two examples of bivariate Biermann sets in $[0,1]^2$ starting from randomly chosen univariate points.



Biermann points ($d = 7$)



Biermann points ($d = 7$)

COMMENT 4 — . One can easily compute interpolants with this kind of points . . . but it gonna take a long way before having them suitably distributed. . .

F. THE WAY OUT...

Something can be said about the Lebesgue constant of Biermann points...

THEOREM 2 (JPC) — . Let K be a compact set in $\mathbb{C}^2$ containing $A \oplus B$. We denote by K_1 (resp., K_2) the projection of K on the axes. We have

$$\Delta(\mathbf{A} \oplus \mathbf{B}|K) \leq 4\binom{n+2}{n} \sum_{i+j\leq n} \Delta\left(\mathbf{A}^{[i]}|K_1\right) \cdot \Delta\left(\mathbf{B}^{[j]})|K_2\right).$$

where $\mathbf{A}^{[i]} = (a_0, \ldots, a_i)$ (and likewise for $\mathbf{B}^{[j]}$).

PROBLEM 3 — . How could me make any practical use of this bound ?

$\longrightarrow$ We need to have points in A such that the Lebesgue constants of all the blocks $A^{[i]}$ can be nicely estimated...

$\longrightarrow$ None of the classical choice of univariate interpolation points, such as for instance, the Chebyshev points, enjoys such property...

$\longrightarrow$ So what we need to find is a **sequence of points** (we add one point when going from d to $d+1$) with nice Lebesgue constants...

FACT 7 — . There is **one** univariate sequence of interpolation points which is available in the litterature...

G. BACK TO LEJA'S POINTS

What is a Leja sequence ?

A Leja sequence for a plane compact set K is a sequence (a_n) in K that satisfies the equation

$$\max_{z\in K}\prod_{i=0}^{d}|z-a_i| = \prod_{i=0}^{d}|a_{d+1}-a_i|, \quad d\geq 0,$$

that is, **the product of the distances to the first $d+1$ points is maximized at $z=a_{d+1}$**.

FACT 8 — . ...Such sequences enjoy many beautiful properties but if we are to produce explicit points we must immediately restrict ourselves to $K=D=\{|z|=1\}$.

The structure of Leja sequences for the unit disc...

We always assume that the first term equals 1.

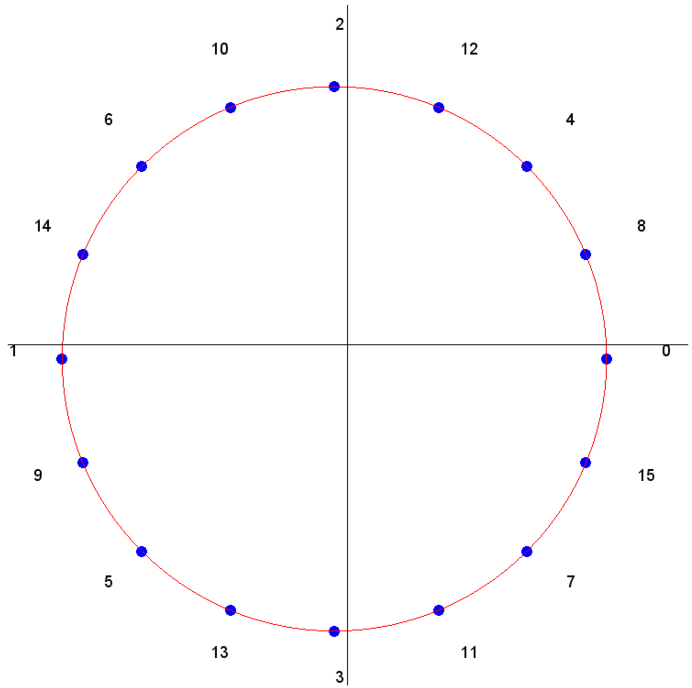
THEOREM 3 (L. Białas-Ciez & JPC) — . Leja sequences for the unit disk obey the following rules.

- The underlying set of a 2^n-Leja section ($=2^n$ first terms) is formed of the 2^n-th roots of unity.
- If $E_{2^{n+1}}$ is a 2^{n+1}-Leja section then there exist a 2^n-th root ρ of -1 and a 2^n-Leja section U_{2^n} such that $E_{2^{n+1}} = (E_{2^n}, \rho U_{2^n})$.

FACT 9 — . There are infinitely many Leja sequences...

Here are the first 16 points of the simplest Leja sequence…

The rule is $\begin{cases} E_2 = (1, -1), \\ E_{2^{n+1}} = (E_{2^n}, e^{i\pi/2^n} E_{2^n}), \quad n \geq 1. \end{cases}$



A lot can be done with such sequences but I want to show you pictures so I must come back in the real world…

H. CONSTRUCTING GOOD REAL POINTS STARTING FROM COMPLEX POINTS...

FACT 10 — . The idea is simple : we shall take the projections on the real axis of Leja points.

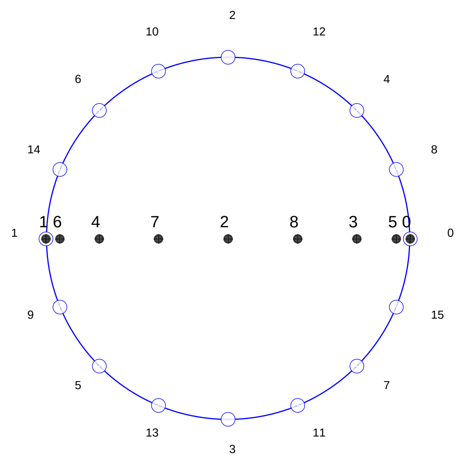
Why should it be a good idea ?

$\longrightarrow$ When we project a complete set of n-th roots of unity, we obtain the Chebyshev-Lobatto points which are known to be excellent interpolation points.

FACT 11 — . There is a slight problem : two points of a Leja sequence have the same projection and we need pairwise distinct points...

DEFINITION 3 — . A sequence X (in $[-1,1]$) is said to be a $\Re$-Leja sequence if there exists a Leja sequence $E = (e_k : k \in \mathbb{N})$ such that X is obtained by eliminating repetitions in $\Re(e_k : k \in \mathbb{N})$. Here, we mean that the entry $\Re(e_j)$ is eliminated whenever there exists $i < j$ such that $\Re(e_j) = \Re(e_i)$. We write $X = X(E)$.

The picture shows the 9 $\Re$-Leja points obtained with 16 Leja points together with their index. Remember that we are not merely looking for a set of point but for an **ordered set** of points.



Here is the precise description of the way we construct a $\Re$-Leja sequence.

THEOREM 4 (Phung VM & JPC) — . A sequence $X = (x_k : k \in \mathbb{N})$ is a $\Re$-Leja sequence if and only if there exists a Leja sequence $E = (e_k : k \in \mathbb{N})$ such that

$$X = (1, -1) \wedge \bigwedge_{j=1}^{\infty} \Re\Big(E(2^j : 2^j + 2^{j-1} - 1)\Big), \tag{0.1}$$

where $E(l : m) := (e_l, e_{l+1}, \ldots, e_m)$.

I. How to study the Lebesgue constant of such a complicated sequence. . . ?

. . . We use the fundamental principle of mathematics : we look for simple parts in a complex whole...

The elementary starting point. . .

Lemma 1 — . Let $N = N_0 \cup \cdots \cup N_{s-1}$ where the N_i form a partition of the finite set $N \subset \mathbb{R}$. We have

$$\ell(N,a;\cdot) = \frac{w_{N\setminus N_i}}{w_{N\setminus N_i}(a)}\ell(N_i,a;\cdot), \quad a \in N_i, \quad i = 0,\dots,s-1, \tag{0.1}$$

where $w_{N\setminus N_i}(z) = \prod_{a\in N\setminus N_i}(z-a)$. Consequently,

$$\Delta(N) \le \sum_{i=0}^{s-1} \max_{x\in K, a\in N_i} \left| \frac{w_{N\setminus N_i}(x)}{w_{N\setminus N_i}(a)} \right| \Delta(N_i), \tag{0.2}$$

where the Lebesgue constants are computed with respect to a compact set K containing N.

A careful study of a $\mathfrak{R}$-Leja enables to identify reasonable subsets A_i whose points look like modified Chebyshev-Lobatto points

$\longrightarrow$ At first sight, it sounds good, one should be able to do something with that...
$\longrightarrow$ But after a while thought, one understands that we are entering a jungle of trigonometric polynomials in the best (or worst) tradition of old fashioned mathematics...

J. A FEW RESULTS AND ILLUSTRATIONS...

The univariate results...

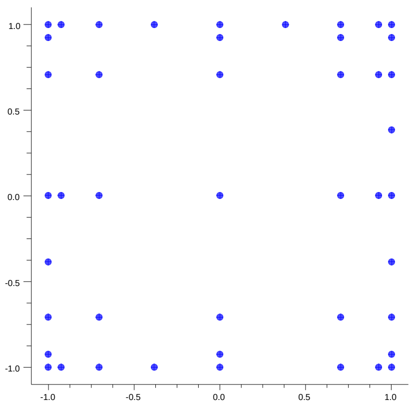
THEOREM 5 (Phung VM & JPC) — . Let E be a Leja sequence for D. As $k \to \infty$, $\Delta(E_k) = O(k \ln k)$.

THEOREM 6 (Phung VM & JPV) — . Let X be a $\Re$-Leja sequence. The Lebesgue constant $\Delta(X_k)$ for the interpolation points $x_0, \ldots, x_{k-1}$ satisfies the following estimate

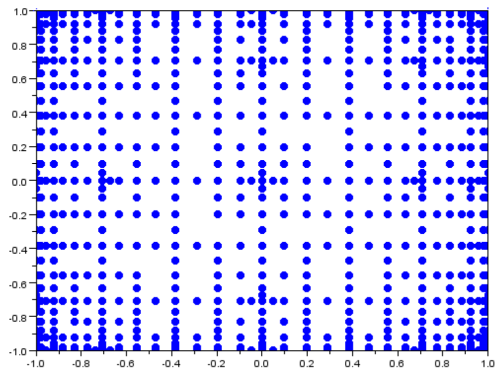
$$\Delta(X_k) = O(k^3 \log k), \quad k \to \infty.$$

Intertwining of the above sequences provide multivariate interpolation points whose Lebesgue constants grow polynomially (the degree may be estimated)...

Here are two examples. ...

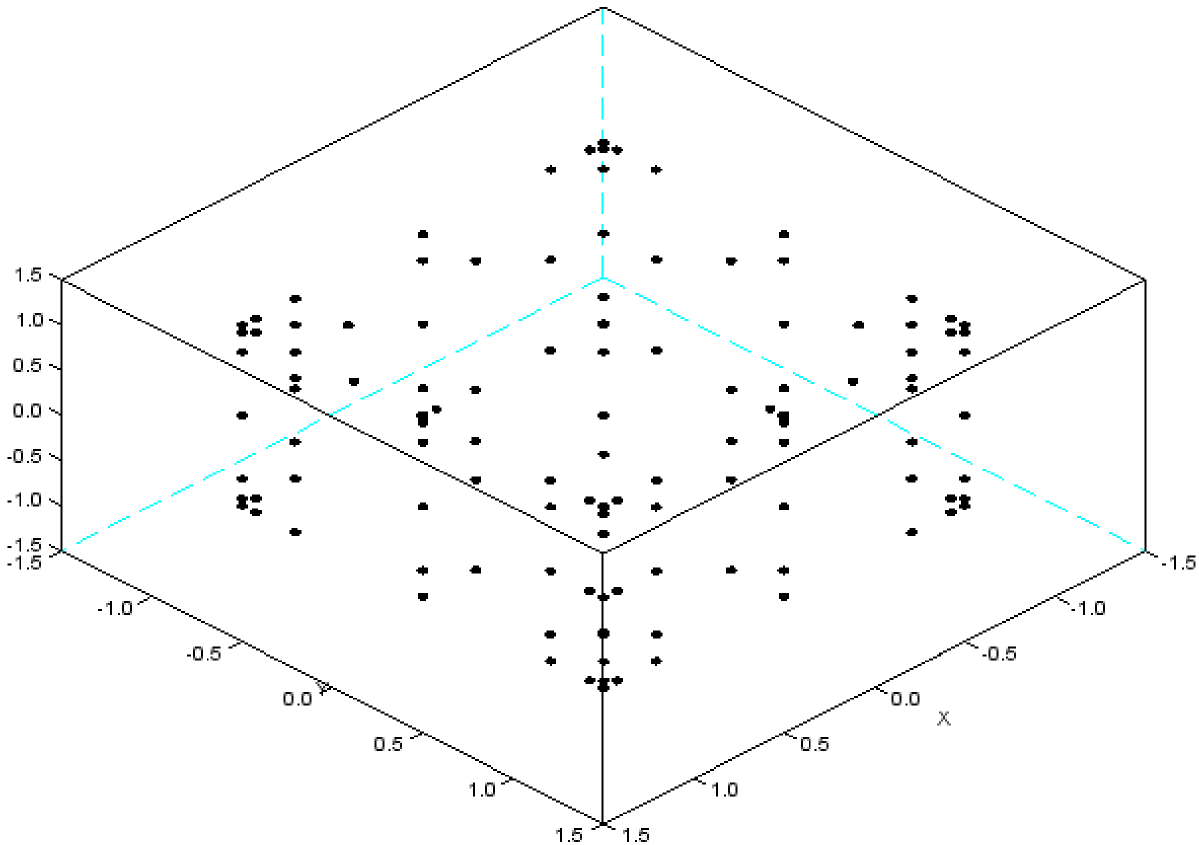


$d = 9$



$d = 40$

We can compute good points in any dimension... But the its harder to visually appreciate...



$d = 4$

K. Limitations and other approaches

1. *Can we expect to find explicit lists of good points for more general compact K ?*
2. *Can we at least use good points in $[-1,1]^N$ for approximation in more general cases ?*
3. *Is there any way of compute lists of good points for more general compact sets ?*
4. *The limitation of deterministic algorithms. . .*
5. *The next step : probabilistic algorithms . . .*

L. A way of computing good points for more general plane compact sets

Working with pseudo Leja sequences

DEFINITION 4 — . Let (a_n) be a sequence of points in K and (M_n) a sequence of subexponential growth with $M_n \geq 1$, $n \in \mathbb{N}$. We say that (a_n) is a *pseudo Leja sequence* of *Edrei growth* M_n if

$$M_n\,|w_n(a_n)| \geq \max_{z\in K} |w_n(z)|, \quad w_n = (\cdot - a_0)\cdots(\cdot - a_{n-1}), \quad n \in \mathbb{N}^\star. \tag{0.1}$$

FACT 12 — . Pseudo Leja sequences discretize the equilibrium measure and are therefore good points for Lagrange interpolation of analytic functions.

Computing Pseudo Leja sequences with the help of admissible meshes

DEFINITION 5 — . We say that a sequence of sets A_n, $n \in \mathbb{N}$, is a *weakly admissible mesh* for K if the following two conditions are satisfied.

(a) A_n is a finite subset of K (whose cardinality grows subexponentially) .

(b) There exists a sequence (M_n) of subexponential growth (as $n \to \infty$) such that, for every polynomial p of degree at most n,

$$\max_{z\in K} |p(z)| \leq M_n \max_{z\in A_n} |p(z)|, \tag{0.2}$$

…Pseudo Leja sequences and admissible meshes

ALGORITHM 7 — . Let (A_n) be a weakly admissible mesh of growth (M_n). We define inductively a sequence (a_n) as follows. We take $a_0 \in K$ and, for $d \geq 1$, we select $a_d \in A_d$ such that

$$|w_d(a_d)| = \max_{z \in A_d} |w_d(z)| \tag{0.3}$$

where, as usual, $w_d(z) = (z - a_0) \cdots (z - a_{d-1})$. Then the sequence (a_n) is a pseudo Leja sequence of Edrei growth (M_n) for K.

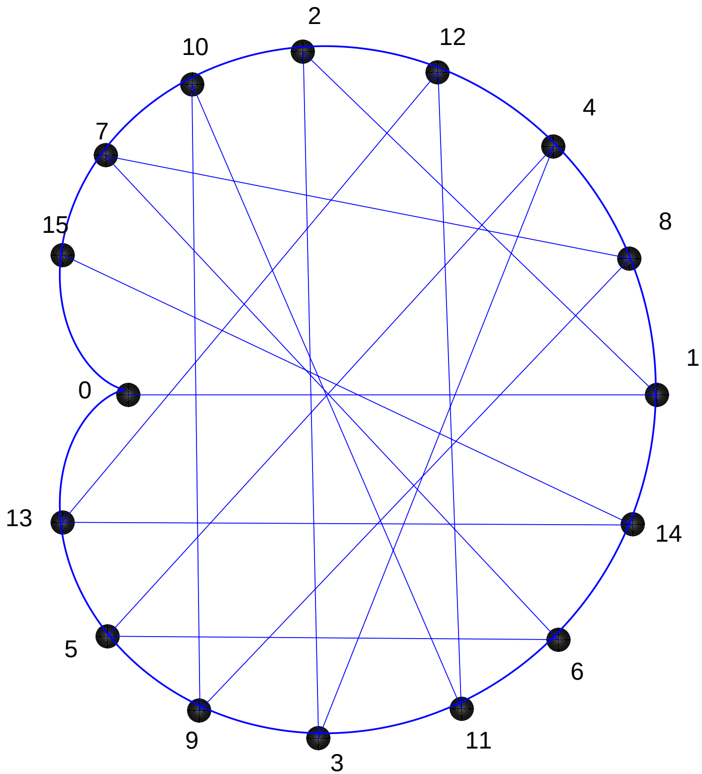
So the problem of computing good points is reduced to that of finding admissible meshes…

There are two techniques…

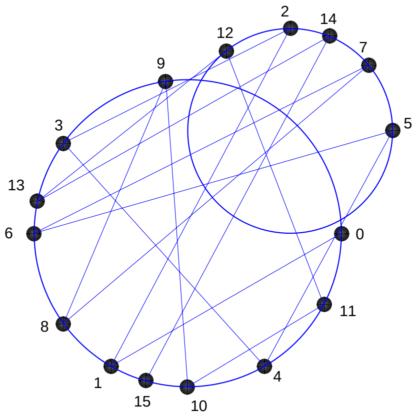
- We use Markov inequality…
- We take unions of points having good Lebesgue constants...

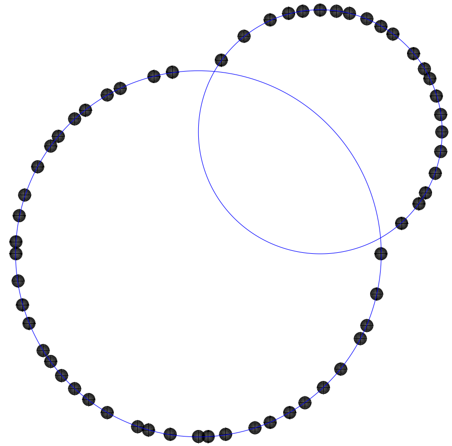
M. EXAMPLES

Pseudo Leja points on a cardioid $\gamma : t \in [0, 2\pi] \to \frac{1}{2}\big(1 + 2\exp(it) + \exp(2it)\big)$



Pseudo Leja points on overlapping disks





Contents

JEAN-PAUL CALVI KRAKÓW 2011