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Approximate Fekete Points

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Abstract

We present an algorithm to compute Approximate Fekete points for polynomial interpolation in one or several real/complex variables.

Fekete Points

$K \subset \mathbb{R}^d$ (or $\mathbb{C}^d$) compact set (notation: $\|f\|_K = \max_{x\in K} |f(x)|$)

$\{p_j\}_{1\le j\le N}$, $N = \dim(\mathbb{P}_n^d(K))$ polynomial basis

$X = \{\xi_1, \dots, \xi_N\} \subset K$ interpolation points

$V(\xi_1, \dots, \xi_N) = [p_j(\xi_i)]$ Vandermonde matrix, $\det(V) \neq 0$

$\Pi_n f(x) = \sum_{j=1}^N f(\xi_j)\,\ell_j(x)$ determinantal Lagrange formula

$$\ell_j(x) = \frac{\det(V(\xi_1,\dots,\xi_{j-1},x,\xi_{j+1},\dots,\xi_N))}{\det(V(\xi_1,\dots,\xi_{j-1},\xi_j,\xi_{j+1},\dots,\xi_N))}\ , \quad \ell_j(\xi_i) = \delta ij$$

Fekete points: $\det(V(\xi_1,\dots,\xi_N))$ is max in $K^N \Rightarrow \|\ell_j\|_K \le 1$
$\Longrightarrow$ bound of the Lebesgue constant (often rather pessimistic)

$$\Lambda_n = \max_{x\in K} \sum_{j=1}^N |\ell_j(x)| \le N = \dim(\mathbb{P}_n^d(K))$$

Fekete points (and Lebesgue constants) are independent of the choice of the basis

Fekete points are analytically known only in few cases:
interval: Gauss-Lobatto points, $\Lambda_n = \mathcal{O}(\log n)$
complex circle: equispaced points, $\Lambda_n = \mathcal{O}(\log n)$
cube: for tensor-product polynomials, $\Lambda_n = \mathcal{O}(\log^d n)$

recent important result:
Fekete points are asymptotically equidistributed with respect to the pluripotential equilibrium measure of K (cf. [1])

essentially open problems:
- asymptotic spacing in the multivariate case (cf. [5])
- efficient computation, even in the univariate complex case (large scale optimization problem in $N \times d$ variables [9])

- idea: extract Fekete points from a discretization of K: but which could be a suitable mesh?

Polynomial Inequalities and Admissible Meshes

Weakly Admissible Mesh (WAM): sequence of discrete subsets $\mathcal{A}_n \subset K$ such that

$$\|p\|_K \le C(\mathcal{A}_n)\|p\|_{\mathcal{A}_n}\ , \quad \forall p \in \mathbb{P}_n^d(K)$$

where $\mathrm{card}(\mathcal{A}_n) \ge N$ and $C(\mathcal{A}_n)$ grow polinomially with n

$C(\mathcal{A}_n)$ bounded: Admissible Mesh (AM)

Properties of (W)AMs (cf. [3, 6]):

- $C(\mathcal{A}_n)$ is invariant under affine mapping
- any sequence of unisolvent interpolation sets whose Lebesgue constant grows polynomially with n is a WAM, $C(\mathcal{A}_n)$ being the Lebesgue constant itself
- a finite union of (W)AMs is a (W)AM for the corresponding union of compacts, $C(\mathcal{A}_n)$ being the maximum of the corresponding constants
- in $\mathbb{C}^d$ a (W)AM of the boundary ∂K is a (W)AM of K (by the maximum principle)
- given a polynomial mapping π_m of degree m, then $\pi_m(\mathcal{A}_{nm})$ is a (W)AM for $\pi_m(K)$ with constants $C(\mathcal{A}_{nm})$
- any K satisfying a Markov polynomial inequality like $\|\nabla p\|_K \le Mn^r\|p\|_K$ has an AM with $\mathcal{O}(n^{rd})$ points

Relevance to polynomial approximation:

- Least Squares polynomial $\mathcal{L}_{\mathcal{A}_n} f$ on a (W)AM, $f \in C(K)$:
$\|f - \mathcal{L}_{\mathcal{A}_n} f\|_K \approx C(\mathcal{A}_n)\sqrt{\mathrm{card}(\mathcal{A}_n)}\ \min\{\|f-p\|_K,\ p \in \mathbb{P}_n^d(K)\}$

- Fekete points extracted from a WAM have a Lebesgue constant $\Lambda_n \le NC(\mathcal{A}_n)$

Approximate Fekete Points

extracting Fekete points from (W)AMs, $\mathcal{A}_n = \{a_1, \dots, a_M\}$

$\Updownarrow$

discrete optimization problem: extracting a maximum volume (determinant) $N \times N$ submatrix from the rectangular $M \times N$ Vandermonde matrix $V(a_1,\dots,a_M) = [p_j(a_i)]$

this is NP-hard: then we look for an approximate solution

Algorithm greedy
(max volume submatrix of $A \in \mathbb{R}^{N\times M}$, $M > N$)

for $j = 1, \dots, N$

- "extract the largest norm column col_{i_j}";
- "remove from every remaining column of A its orthogonal projection onto col_{i_j}";

end;

this algorithm can be easily implemented by the well known QR factorization with column pivoting by Businger and Golub (1965), applied to $A = V^t$ (in Matlab/Octave, simply via the standard "backslash" linear solver!)

Key asymptotic result (cf. [3]): the Approximate Fekete points extracted from a (W)AM by the greedy algorithm have the same asymptotic behavior of the true Fekete points

$$\text{discrete measures } \frac{1}{N}\sum_{j=1}^N \delta_{\xi_j} \xrightarrow{weak*} \text{equilibrium measure of } K$$

Numerical Algorithm

Algorithm AFP
(Approximate Fekete Points by iterative refinement)

- take a (Weakly) Admissible Mesh $\mathcal{A}_n = (a_1, \dots, a_M) \subset K$
- $V_0 = V(a_1, \dots, a_M)$; $P_0 = I$;
- for $k = 0, \dots, s-1$
  $V_k = Q_k R_k$; $U_k = \mathrm{inv}(R_k)$;
  $V_{k+1} = V_k U_k$; $P_{k+1} = P_k U_k$;
  end ;
- $A = V_s^t$; $b = (1, \dots, 1)^t$; (b is irrelevant in practice)
- $w = A \backslash b$; (this implements the greedy algorithm)
- $ind = \mathrm{find}(w \neq 0)$; $X = \mathcal{A}_n(ind)$;

main feature: change with the nearly orthogonal basis $(q_1, \dots, q_N) = (p_1, \dots, p_N)P_s$ with respect to the discrete inner product $(f, g) = \sum f(a_i)\overline{g}(a_i)$

tries to overcome possible numerical rank-deficiency and severe ill-conditioning arising with nonorthogonal bases

Approximate Fekete Points in One Variable

Figure 1. $N = 31$ Approximate Fekete points (deg $n = 30$) from Admissible Meshes in: one interval, two and three disjoint intervals

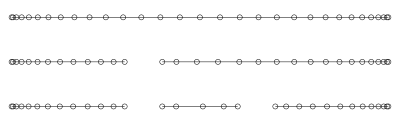
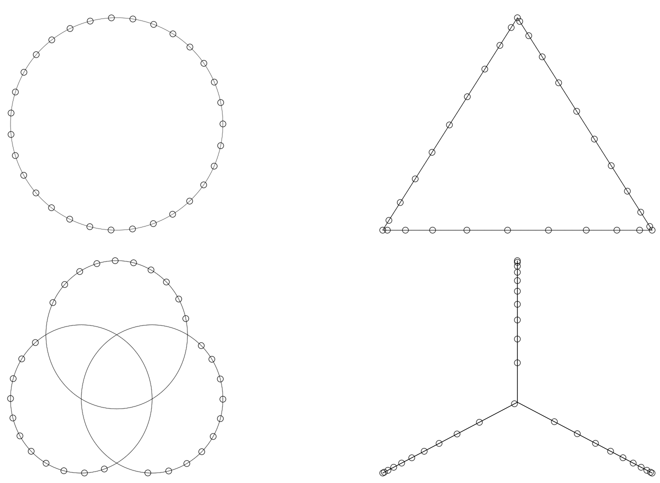


Figure 2. As above for some compacts in the complex plane



Approximate Fekete Points in Two Variables

Admissible Meshes on 2-dimensional compacts: $\mathcal{O}(n^4)$ points

geometric WAMs (Weakly Admissible Meshes): obtained by a suitable transformation, much lower cardinality!

example: Duffy quadratic transformation of the Padua interpolation points of degree $2n$ (cf. [2]) from the square onto the triangle: $\mathcal{O}(n^2)$ points, $C(\mathcal{A}_n) = \mathcal{O}(\log^2 2n)$

WAMs on polygons by triangulation and finite union

Figure 3. 861 Padua points of deg $2n = 40$ in the square and the corresponding geometric WAM (dots) with $N = 231$ Approximate Fekete points (circles) of deg $n = 20$ for the triangle

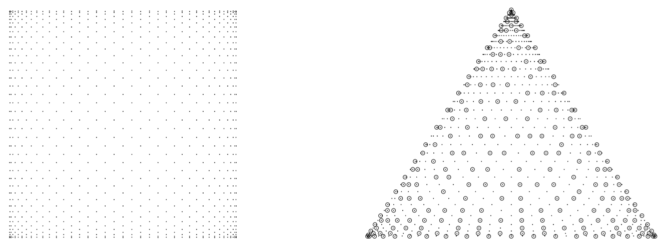
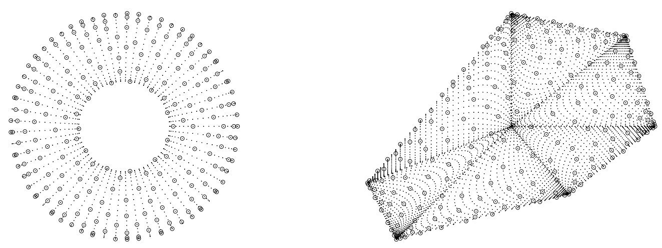


Figure 4. geometric WAMs (dots) with $N = 231$ Approximate Fekete points (circles) of deg $n = 20$ for an annulus and a polygon



Lebesgue Constants

Table 1. Numerically estimated Lebesgue constants of interpolation points in some 1-dimensional real and complex compacts (Figs. 1-2)

points	$n = 10$	20	30	40	50	60
	$N = 11$	21	31	41	51	61
equisp intv	29.9	1e+4	6e+6	4e+8	7e+9	1e+10
Fekete intv	2.2	2.6	2.9	3.0	3.2	3.3
AFP intv	2.3	2.8	3.1	3.4	3.6	3.8
AFP 2intvs	3.1	6.3	7.1	7.6	7.5	7.2
AFP 3intvs	4.2	7.9	12.6	6.3	5.8	5.3
AFP disk	2.7	3.0	3.3	3.4	3.5	3.7
AFP triangle	3.2	6.2	5.2	4.8	9.6	6.1
AFP 3disks	5.1	3.0	7.6	10.6	3.8	8.3
AFP 3branches	4.7	3.5	3.8	8.3	5.0	4.8

Table 2. As above in some 2-dimensional real compacts (Figs. 3-4)

points	$n = 6$	10	14	18	22	26	30
	$N = 28$	66	120	190	276	378	496
Padua square	5.4	6.9	8.0	8.8	9.5	10.2	10.7
Fekete triangle [9]	4.2	7.8	9.7	13.5	*	*	*
AFP triangle	7.1	14.9	24.8	35.4	72.1	70.2	89.5
AFP annulus	8.3	17.7	28.3	35.9	55.9	62.7	93.2
AFP polygon	6.3	15.6	22.8	26.3	46.7	87.5	75.9

Developments and Applications

- algebraic cubature: b = moments in Alg. AFP $\Rightarrow$ w = weights
- weighted interpolation: prescribed poles, digital filters, ...
- three-dimensional instances: cube, ball, tetrahedron, ...
- numerical PDEs: spectral and high order methods, collocation, discrete least squares (promising results in [7, 10]), ...

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